Graded ideals whose quotient rings are Gorenstein

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1. Introduction

Question 1.1

Let A be a Gorenstein ring with $\dim A > 0$. How many non-principal ideals I of A such that $\mathrm{ht}_A I = 1$ and A/I is Gorenstein exist?

Let

- $(A, \mathfrak{m})$ a CM local ring with $d = \dim A \geq 0$
- I an $\mathfrak{m}$-primary ideal of A.

Recall that

I is Ulrich $\overset{\text{def}}{\Longleftrightarrow}$ (1) $\mathrm{gr}_I(A) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is CM with $\mathrm{a}(\mathrm{gr}_I(A)) = 1 - d$
(2) I/I^2 is A/I-free. (Goto-Ozeki-Takahashi-Watanabe-Yoshida)

When I contains a parameter ideal Q as a reduction (i.e., $I^{r+1} = QI^r$ for $\exists r \geq 0$),

$$I \text{ is Ulrich} \iff I \neq Q,\ I^2 = QI, \text{ and } I/Q \text{ is } A/I\text{-free}$$

because of $0 \to Q/QI \to I/I^2 \to I/Q \to 0$.

Then $I/Q \cong (A/I)^{\oplus(n-d)}$, where $n = \mu_A(I)$. Hence

$$(n-d)\cdot \mathrm{r}(A/I) = \mathrm{r}_A(I/Q) \le \mathrm{r}(A/Q) = \mathrm{r}(A)$$

so that $d + 1 \le \mu_A(I) \le d + \mathrm{r}(A)$.

Fact 1.2 (GOTWY, Goto-Takahashi-T)

$$A \textit{ is Gorenstein} \iff \mu_A(I) = d+1 \textit{ and } A/I \textit{ is Gorenstein}$$

provided that Ulrich ideal I exists.

Question 1.3

Let $R = k[H]$ be a semigroup ring of a numerical semigroup H over a field k. Suppose R is Gorenstein. Can we estimate

$$\#\{I \mid I \text{ is a graded ideal of } R,\ R/I \text{ is Gorenstein, and } \mu_R(I) \ge 2\}?$$

2. Main theorem

- $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$
- H a numerical semigroup, i.e., a submonoid of $\mathbb{N}$ with $\#(\mathbb{N} \setminus H) < \infty$
- $\mathrm{c}(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for } \forall m \in \mathbb{Z} \text{ with } m \geq n\}$
- k a field
- $R = k[H] = k[t^h \mid h \in H] \subseteq k[t]$
- $\mathcal{X}_R = \{I \mid I \text{ is a graded ideal of } R,\ R/I \text{ is Gorenstein, and } \mu_R(I) \geq 2\}$

Note that

- R is a CM graded domain with $\dim R = 1$, $\mathrm{a}(R) = \mathrm{c}(H) - 1$, and $\overline{R} = k[t]$.
- R is Gorenstein $\iff$ H is symmetric (Herzog-Kunz)

$$\overset{\text{def}}{\iff} \#\{n \in H \mid n < \mathrm{c}(H)\} = \#(\mathbb{N} \setminus H)$$

$$\iff \#(\mathbb{N} \setminus H) = \frac{\mathrm{c}(H)}{2}.$$

Recall $\mathcal{X}_R = \{I \mid I \text{ is a graded ideal of } R,\ R/I \text{ is Gorenstein, and } \mu_R(I) \geq 2\}$.
Note that $a = \mathrm{a}(R) \neq \mathrm{a}(R/I)$ for $\forall I \in \mathcal{X}_R$.

Theorem 2.1 (Main theorem)

Suppose that R is Gorenstein. Then the following assertions hold true.

(1) $\mathbb{N} \setminus H \overset{1:1}{\longleftrightarrow} \{I \in \mathcal{X}_R \mid \mathrm{a}(R/I) < a\}, \ m \mapsto R :_R t^m$.

(2) $\{I \in \mathcal{X}_R \mid \mathrm{a}(R/I) > a\} \overset{1:1}{\longleftrightarrow} \{I \in \mathcal{X}_R \mid \mathrm{a}(R/I) < a\}, \ I \mapsto t^{a-\mathrm{a}(R/I)}I$.

(3) $\mathcal{X}_R = \{R :_R t^m, \ t^m(R :_R t^m) \mid m \in \mathbb{N} \setminus H\}$.

In particular, $\#\mathcal{X}_R = \mathrm{c}(H)$.

Remark 2.2

There exists a one-dimensional local Gorenstein numerical semigroup ring A with infinite residue class field (e.g., $\mathbb{Q}[[t^3, t^7]]$, $\mathbb{C}[[t^4, t^5, t^6]]$) admitting infinitely many two-generated Ulrich ideals.

Let

- $(A, \mathfrak{m})$ a Gorenstein complete local domain with $\dim A = 1$ s.t. $A/\mathfrak{m}$ is algebraically closed
- $v(A) = \{\mathrm{o}(f) \mid 0 \neq f \in A\}$ the value semigroup of A
- $\mathfrak{n}$ the maximal ideal of the DVR $\overline{A}$.

For $\forall \ell \in \mathbb{Z}$, we set $F_\ell = \mathfrak{n}^\ell \cap A$. Then $\mathcal{F} = \{F_\ell\}_{\ell \in \mathbb{Z}}$ is a filtration of ideals in A.

Define

$$G = G(\mathcal{F}) = \bigoplus_{\ell \geq 0} F_\ell / F_{\ell+1} \cong (A/\mathfrak{m})[v(A)]$$

because, for each $\ell \geq 0$, $G_\ell \neq (0)$ if and only if $\ell \in v(A)$.

Corollary 2.3

The equality

$$\# \{I \mid I \text{ is a graded ideal of } G,\ G/I \text{ is Gorenstein, and } \mu_G(I) \geq 2\} = \mathrm{c}(v(A))$$

holds.

3. Examples

Theorem 2.1 (Main theorem)

If $R = k[H]$ *is Gorenstein, then* $\mathcal{X}_R = \{R :_R t^m,\ t^m(R :_R t^m) \mid m \in \mathbb{N} \setminus H\}$.

Example 3.1

(1) Let $H = \langle 2, 2\ell+1 \rangle$ $(\ell \geq 1)$. Then $\mathrm{c}(H) = 2\ell$, and

$$\mathcal{X}_R = \{(t^2, t^{2\ell+1}), (t^4, t^{2\ell+1}), \dots, (t^{2\ell}, t^{2\ell+1}),$$
$$(t^{2\ell+1}, t^{4\ell}), (t^{2\ell+1}, t^{4\ell-2}), \dots, (t^{2\ell+1}, t^{2\ell+2})\}.$$

Indeed, since $\mathbb{N} \setminus H = \{1, 3, 5, \dots, 2\ell-1\}$, we have

$$R :_R t^{2\ell-1} = (t^2, t^{2\ell+1}), \quad t^{2\ell-1}(R :_R t^{2\ell-1}) = (t^{2\ell+1}, t^{4\ell})$$
$$R :_R t^{2\ell-3} = (t^4, t^{2\ell+1}), \quad t^{2\ell-3}(R :_R t^{2\ell-3}) = (t^{2\ell+1}, t^{4\ell-2})$$
$$\vdots$$
$$R :_R t = (t^{2\ell}, t^{2\ell+1}), \quad t(R :_R t) = (t^{2\ell+1}, t^{2\ell+2}).$$

0	1
2	3
4	5
$\vdots$	$\vdots$
$2\ell-2$	$2\ell-1$
2ℓ	$2\ell+1$

Example 3.2

(2) Let $H = \langle 3, 4 \rangle$. Then $\mathrm{c}(H) = 6$ and

$$\mathcal{X}_R = \{(t^3, t^4), (t^4, t^6), (t^3, t^8), (t^8, t^9), (t^6, t^8), (t^4, t^9)\}.$$

(3) Let $H = \langle 3, 5 \rangle$. Then $\mathrm{c}(H) = 8$ and

$$\mathcal{X}_R = \{(t^3, t^5), (t^5, t^6), (t^3, t^{10}), (t^5, t^9), (t^{10}, t^{12}), (t^9, t^{10}), (t^5, t^{12}), (t^6, t^{10})\}.$$

(4) Let $H = \langle n, n+1, \ldots, 2n-2 \rangle$ $(n \geq 4)$. Then $\mathrm{c}(H) = 2n$ and

$$\begin{aligned}\mathcal{X}_R = & \{(t^n, t^{n+1}, \ldots, t^{2n-2}), (t^{n+1}, t^{n+2}, \ldots, t^{2n-2}, t^{2n})\} \\ & \bigcup \{(t^n, t^{n+1}, \ldots, t^{n+i-1}, t^{n+i+1}, \ldots, t^{2n-2}) \mid 1 \leq i \leq n-2\} \\ & \bigcup \{(t^{3n-1}, t^{3n}, \ldots, t^{4n-3}), (t^{2n}, t^{2n+1}, \ldots, t^{3n-3}, t^{3n-1})\} \\ & \bigcup \{(t^{2n-i-1}, t^{2n-i}, \ldots, t^{2n-2}, t^{2n}, \ldots, t^{3n-i-3}) \mid 1 \leq i \leq n-2\}.\end{aligned}$$

Thank you for your attention.